

GOLDBACH'S PROBLEM IN PRIMES WITH BINARY EXPANSIONS OF A SPECIAL FORM

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ABSTRACT. Let \mathbb{N}_0 be a class of natural numbers whose binary expansions contain even numbers of ones. Goldbach's problem in numbers of class \mathbb{N}_0 is solved.

1. Introduction

Let $n = e_0 + e_1 2 + \dots + e_k 2^k$ be a binary expansion of a natural number n , ($e_j = 0, 1$). Let \mathbb{N}_0 be a set of natural numbers whose binary expansions have an even number of ones, $\mathbb{N}_1 = \mathbb{N} \setminus \mathbb{N}_0$. Let

$$\varepsilon(n) = \begin{cases} 1, & \text{if } n \in \mathbb{N}_0; \\ -1, & \text{if } n \in \mathbb{N}_1. \end{cases}$$

In 1968, A.O. Gelfond [1] proved that numbers from the sets \mathbb{N}_0 and \mathbb{N}_1 are regularly distributed in arithmetical progressions.

In 1991, The author got [2] the asymptotical formula for the sum

$$\sum_{n \leq x, n \in \mathbb{N}_0} \tau(n)$$

and so solved Dirichlet divisors problem in the numbers of class \mathbb{N}_0 .

In 2010, C. Mauduit and J. Rivat [3] proved in particular that the densities of sets of primes of the classes \mathbb{N}_0 and \mathbb{N}_1 are equal to each other. B. Green gave another proof of this fact [4]. These papers are based on estimates of exponential sums of a special type, which, by the force and by methods of proofs, are variants of estimate, derived by the author in 1991, of the integral of modulus of a trigonometric sum of the special type [2].

In this paper the ternary Goldbach problem in prime numbers of the set \mathbb{N}_0 is solved.

The main results are contained in the following theorems.

Theorem 1. *Let α be an arbitrary real number. There exists an absolute constant $\varkappa > 0$ such that*

$$S = \sum_{n \leq X} \varepsilon(n) \Lambda(n) e^{2\pi i \alpha n} = O(X^{1-\varkappa}).$$

The constant in sign O is absolute.

Theorem 2. *Let $J(N)$ be the number of representations of odd N by sum of three primes, and $J_0(N)$ be the number of representations of odd N by sum of three primes from the set \mathbb{N}_0 .*

Then the equality

$$J_0(N) = \frac{1}{8} J(N) (1 + O(N^{-\varkappa} \ln N)),$$

holds, where $\varkappa > 0$ is a constant from theorem 1.

Key words and phrases. Goldbach's problem, Gelfond's problem, binary expansion, sequence of natural numbers, trigonometric sum, complex-valued function, inequality of the large sieve.

2. Auxiliary lemmas

Lemma 1. *Let*

$$\alpha = \frac{a}{q} + \frac{\theta}{q^2}, \quad (a, q) = 1, \quad q \geq 1, \quad |\theta| \leq 1.$$

Then for any $\beta \in \mathbb{R}$, $U > 0$, $P \geq 1$ we have

$$\sum_{x=1}^P \min(U, \|\alpha x + \beta\|^{-1}) \leq 6 \left(\frac{P}{q} + 1 \right) (U + q \log q).$$

Proof see in [5, chapter 4].

Lemma 2. (A.O. Gelfond) *Let $Q \in \mathbb{N}$. The inequality*

$$\left| \prod_{r=0}^{2Q-1} \left(1 - e^{2\pi i \alpha 2^r} \right) \right| \leq \frac{2}{\sqrt{3}} 2^{2Q\lambda},$$

holds, where $\lambda = \frac{\ln 3}{\ln 4} = 0,7924812\dots$

Proof see in [1].

Corollary 1. *For any $\alpha \in \mathbb{R}$ the estimate*

$$\left| \sum_{n \leq X} \varepsilon(n) e^{2\pi i \alpha n} \right| = O(X^\lambda \ln X).$$

holds.

Proof. Define natural number Q with inequalities

$$2^{2(Q-1)} < X + 1 \leq 2^{2Q}.$$

Then

$$\begin{aligned} \left| \sum_{n \leq X} \varepsilon(n) e^{2\pi i \alpha n} \right| &= \left| \sum_{n < 2^{2Q}} \varepsilon(n) e^{2\pi i \alpha n} \sum_{n_1 \leq X} \frac{1}{2^{2Q}} \sum_{l=1}^{2^{2Q}} e^{2\pi i \frac{(n-n_1)l}{2^{2Q}}} \right| \leq \\ &\leq 2^{-2Q} \sum_{l=1}^{2^{2Q}} \left| \sum_{n < 2^{2Q}} \varepsilon(n) e^{2\pi i (\alpha + l 2^{-2Q}) n} \right| \left| \sum_{n_1 \leq X} e^{-2\pi i n_1 l 2^{-2Q}} \right|. \end{aligned}$$

Furthermore, it follows from inequality

$$\sum_{n < 2^{2Q}} \varepsilon(n) e^{2\pi i (\alpha + l 2^{-2Q}) n} = \prod_{r=0}^{2Q-1} \left(1 - e^{2\pi i (\alpha + l 2^{-2Q}) 2^r} \right)$$

and lemma 2 that

$$\left| \sum_{n \leq X} \varepsilon(n) e^{2\pi i \alpha n} \right| \ll X^\lambda 2^{-2Q} \sum_{l=1}^{2^{2Q}} \min(X, \|l 2^{-2Q}\|^{-1}).$$

From this and Lemma 1 we have Corollary 1.

Lemma 3. (Gallagher). *Let $S(t)$ be a complex valued function with continuous first derivative on $[t_0, t_k]$ and, $t_0 < t_1 < \dots < t_{k-1} < t_k$.*

Then, assuming that $\delta = \min_{0 \leq r < k} (t_{r+1} - t_r)$, we have

$$\sum_{r=1}^k |S(t_r)| \leq \frac{1}{\delta} \int_{t_0}^{t_k} |S(t)| dt + \frac{1}{2} \int_{t_0}^{t_k} |S'(t)| dt.$$

Proof see in [6, chapter 1].

3. The main lemma and its corollaries

Lemma 4. *Let $Q \in \mathbb{N}$,*

$$S_Q(a) = \prod_{r=0}^{2Q-1} (1 - e^{2\pi i a 2^r}).$$

The inequality

$$\int_0^1 |S_Q(\alpha)| d\alpha \leq 2^{Q\theta_0},$$

holds, where $\theta_0 = \log_2 \sqrt{2 + \sqrt{2}} = 0,88577 \dots$

Proof see in [2].

Corollary 2. *The inequality*

$$\int_0^1 \left| \sum_{n \leq X} \varepsilon(n) e^{2\pi i \alpha n} \right| d\alpha \leq X^{\theta/2} \ln X.$$

holds

Proof. Let $Q \in \mathbb{N}$, $2^{2(Q-1)} < X + 1 \leq 2^{2Q}$

Then

$$\int_0^1 \left| \sum_{n \leq X} \varepsilon(n) e^{2\pi i \alpha n} \right| d\alpha \ll 2^{-2Q} \sum_{l=1}^{2Q} \min(X, \|l2^{-2Q}\|^{-1}) \int_0^1 |S_Q(\alpha + l2^{-2Q})| d\alpha.$$

Since $S_Q(\alpha)$ is a periodic function of t with period 1,

$$\int_0^1 |S_Q(\alpha + l2^{-2Q})| d\alpha = \int_0^1 |S_Q(\alpha)| d\alpha.$$

By lemma 1,

$$2^{-2Q} \sum_{l=1}^{2Q} \min(X, \|l2^{-2Q}\|^{-1}) \ll \ln X.$$

The assertion of Corollary 2 follows from Lemma 4.

Corollary 3. $k \in \mathbb{N}$. *The estimate*

$$2^{-k} \sum_{r=0}^{2^k-1} \left| \sum_{x=0}^{2^k-1} \varepsilon(x) e^{2\pi i x \frac{r}{2^k}} \right| \ll k 2^{\theta k}$$

holds.

Proof. Applying Lemma 3, putting it

$$S(t) = \sum_{x=0}^{2^k-1} \varepsilon(x) e^{2\pi i t x}, \quad t_r = \frac{r}{2^k}.$$

Then

$$S'(t) = 2\pi i \sum_{x=0}^{2^k-1} x\varepsilon(x)e^{2\pi itx},$$

$$\delta = 2^{-k}.$$

Then

$$\begin{aligned} 2^{-k} \sum_{r=0}^{2^k-1} \left| \sum_{x=0}^{2^k-1} \varepsilon(x) e^{2\pi i \frac{rx}{2^k}} \right| &= 2^{-k} \sum_{r=0}^{2^k-1} |S(t_r)| \leq \\ &\leq \int_0^1 |S(t)| dt + 2^{-k} \int_0^1 |S'(t)| dt. \end{aligned}$$

Apply Abel's transform to $S'(t)$:

$$\begin{aligned} 2^{-k} \int_0^1 |S'(t)| dt &\ll 2^{-k} \int_0^1 2^k \left| \sum_{x=0}^{2^k-1} \varepsilon(x) e^{2\pi ixt} \right| dt + \\ &+ 2^{-k} \int_0^1 \int_0^{2^k-1} \left(\left| \sum_{x \leq u} \varepsilon(x) e^{2\pi ixt} \right| \right) du \ll \int_0^1 \left| \sum_{x \leq u_0} \varepsilon(x) e^{2\pi ixt} \right| dt, \end{aligned}$$

where u_0 is a number from the segment $[0, 2^k - 1]$ such that the last integral reaches its maximum. Now, Corollary 3 follows from Corollary 2.

Corollary 4. *Let k, t be integers, $0 \leq t \leq k$. Suppose that the inequality $m \in \mathbb{N}$ holds. Then*

$$\hat{\varepsilon}_m(r) = 2^{-m} \sum_{x=0}^{2^m-1} \varepsilon(m) e^{-2\pi i \frac{rx}{2^m}}.$$

Let a is any number of the segment $[0, 2^k - 1]$. Then

$$\sum_{\substack{r=0 \\ r \equiv a \pmod{2^t}}}^{2^k-1} |\hat{\varepsilon}_k(r)| \ll 2^{(0,5-c)(k-t)} |\hat{\varepsilon}_t(a)| k,$$

where $c = 1/2 - \theta_0/2$, θ_0 is a number from lemma 4.

Proof. By definition we have

$$\begin{aligned} \sum_{\substack{r=0 \\ r \equiv a \pmod{2^t}}}^{2^k-1} |\hat{\varepsilon}_k(r)| &= 2^{-k} \sum_{r \pmod{2^{k-t}}} \left| \sum_{x=0}^{2^k-1} \varepsilon(x) e^{2\pi i \frac{a+2^t r}{2^k} x} \right| = \\ &= 2^{-k} \sum_{r \pmod{2^{k-t}}} \left| \sum_{x=0}^{2^{k-t}-1} \sum_{y=0}^{2^t-1} \varepsilon(x + 2^{k-t} y) e^{2\pi i \frac{a+2^t r}{2^k} (x + 2^{k-t} y)} \right|. \end{aligned}$$

Since $\varepsilon(x + 2^{k-t} y) = \varepsilon(x)\varepsilon(y)$, we have

$$\sum_{\substack{r=0 \\ r \equiv a \pmod{2^t}}}^{2^k-1} |\hat{\varepsilon}_k(r)| = 2^{-(k-t)} \sum_{r \pmod{2^{k-t}}} \left| \sum_{x=0}^{2^{k-t}-1} \varepsilon(x) e^{2\pi i \left(\frac{r}{2^{k-t}} + \frac{a}{2^k} \right) x} \right| \left| 2^{-t} \sum_{y=0}^{2^t-1} \varepsilon(y) e^{2\pi i \frac{ay}{t}} \right| =$$

$$= 2^{-(k-t)} \sum_{r \pmod{2^{k-t}}} \left| \sum_{x=0}^{2^{k-t}-1} \varepsilon(x) e^{2\pi i \left(\frac{r}{2^{k-t}} + \frac{a}{2^k} \right) x} \right| |\varepsilon_t(a)|.$$

The sum on the right side of this inequality is estimated in the same way as a similar amount of Corollary 3.

4. Proof of Theorem 1

Using Vaughan's identity (see, eg [5, chapter 3, problem 9]), with $u = X^{0,1}$:

$$S = \sum_{n \leq X} \Lambda(n) \varepsilon(n) = W_1 - W_2 - W_3 + O(u \ln u),$$

where

$$\begin{aligned} W_1 &= \sum_{d \leq u} \mu(d) \sum_{n \leq Xd^{-1}} \varepsilon(dn) e^{2\pi i \alpha d n} \ln n, \\ W_2 &= \sum_{d \leq u} \mu(d) \sum_{n \leq u} \Lambda(n) \sum_{dnr \leq X} \varepsilon(dnr) e^{2\pi i \alpha dnr}, \\ W_3 &= \sum_{u < m \leq Xu^{-1}} a_m \sum_{u < n \leq Xm^{-1}} \Lambda(n) \varepsilon(mn) e^{2\pi i \alpha mn}, \\ a_m &= \sum_{d|m, d \leq u} \mu(d). \end{aligned}$$

Sums W_1 and W_2 are estimated in the same way. Estimate W_1 .

Fix $d \leq u$. Apply to the inner sum, which we denote $S_1(d)$, the Abel transform, we obtain:

$$|S_1(d)| \ll \left| \sum_{dn \leq u_0} \varepsilon(dn) e^{2\pi i \alpha d n} \right| \log X,$$

where u_0 is a number not exceeding X .

Furthermore,

$$\begin{aligned} \sum_{dn \leq u_0} \varepsilon(dn) e^{2\pi i \alpha d n} &= \sum_{m \leq u_0} \varepsilon(m) e^{2\pi i \alpha m} \frac{1}{d} \sum_{b=0}^{d-1} e^{2\pi i \frac{bm}{d}}, \\ \left| \sum_{dn \leq u_0} \varepsilon(dn) e^{2\pi i \alpha d n} \right| &\leq \frac{1}{d} \sum_{b=0}^{d-1} \left| \sum_{m \leq u_0} \varepsilon(m) e^{2\pi i \left(\alpha + \frac{b}{d} \right) m} \right|. \end{aligned}$$

The sum over m estimate by Corollary 1:

$$\left| \sum_{m \leq u_0} \varepsilon(m) e^{2\pi i \left(\alpha + \frac{b}{d} \right) m} \right| \ll X^\lambda \ln X,$$

where $\lambda = 0,792\dots$

Thus, for any $d \leq u$ we have

$$|S_1(d)| \ll X^\lambda \ln X,$$

therefore,

$$|W_1| \ll u X^\lambda \ln X.$$

Similarly, we arrive at the estimate

$$|W_2| \ll u^2 X^\lambda \ln X.$$

The parameter u is chosen so that

$$|W_1| \ll X^{1-\varkappa_1} \quad |W_2| \ll X^{1-\varkappa_1},$$

where $\varkappa_1 > 0$ is an absolute constant.

We now estimate W_3 . Divide the interval of summation over m in $O(\ln X)$ intervals of the form $(\frac{M}{2}, M]$, where $u < \frac{M}{2} < M \leq \frac{X}{u}$; one of these intervals may be incomplete. So, we have:

$$W_3 = \sum_M^{\ln X} W_3(M),$$

where

$$W_3(M) = \sum_{M/2 < m \leq M_1} a_m \sum_{u < n \leq X/m} \Lambda(n) \varepsilon(mn) e^{2\pi i \alpha mn},$$

where $M/2 < M_1 \leq M$.

Furthermore,

$$\begin{aligned} W_3(M) &= \sum_{M/2 < m \leq M_1} a_m \sum_{u < n \leq X/M_1} \Lambda(n) \varepsilon(mn) e^{2\pi i \alpha mn} + \\ &+ \sum_{M/2 < m \leq M_1} a_m \sum_{X/M_1 < n \leq X/m} \Lambda(n) \varepsilon(mn) e^{2\pi i \alpha mn}. \end{aligned}$$

Splitting the interval of summation over n in $O(\log X)$ intervals of the form $(\frac{N}{2}, N_1]$, where $\frac{N}{2} < N_1 \leq N$, $u < N \leq \frac{X}{M_1}$, we arrive at the inequality

$$|W_3| \ll |W_3(M, N)| \ln^2 X,$$

where

$$W_3(M, N) = \sum_{M/2 < m \leq M_1} a_m \sum_{N/2 < n \leq N_1} \Lambda(n) \varepsilon(mn) e^{2\pi i \alpha mn},$$

where $u < M/2 < M_1 \leq M \leq Xu^{-1}$, $u < N/2 < N_1 \leq N \leq Xu^{-1}$; it may be that $N_1 = X/m$.

Without loss of generality, we assume that $M \leq N$.

Using the fact that $|a_m| \leq \tau(m) \ll m^\varepsilon$ we apply the Cauchy inequality:

$$|W_3(M, N)|^2 \ll M^{1+\varepsilon} \sum_{M/2 < m \leq M_1} \left| \sum_{N/2 < n \leq N_1} \Lambda(n) \varepsilon(mn) e^{2\pi i \alpha mn} \right|^2.$$

Let $H = [X^\rho]$, where $0 < \rho < 10^{-3}$ - a small parameter to be chosen later. We apply van der Corput' inequality (see, eg, [4], [5, 1]):

$$\begin{aligned} &\left| \sum_{N/2 < n \leq N_1} \Lambda(n) \varepsilon(mn) e^{2\pi i \alpha mn} \right|^2 \ll \frac{N}{H} \sum_{|h| \leq H} \left(1 - \frac{|h|}{H} \right) \times \\ &\times \sum_{\substack{N/2 < n \leq N_1, \\ N/2 < n+h \leq N_1}} \Lambda(n) \Lambda(n+h) \varepsilon(mn) \varepsilon(mn+mh) e^{2\pi i \alpha mn} e^{-2\pi i \alpha m(n+h)}. \end{aligned}$$

The contribution of $h = 0$ is estimated as $O\left(\frac{N^2}{H}\right)$, so

$$|W_3(M, N)|^2 \ll \frac{X^{1+\varepsilon}}{H} \sum_{h=1}^H \sum_{N/2 < n \leq N_1} \left| \sum_{M/2 < m \leq M_1} \varepsilon(mn) \varepsilon(mn + mh) e^{-2\pi i \alpha m(n+h)} \right| + \frac{X^{2+\varepsilon}}{H}.$$

Fix $h \in [1, H]$. Now it is sufficient to prove that

$$W_4(M, N) = \sum_{N/2 < n \leq N} \left| \sum_{M/2 < m \leq M_1} \varepsilon(mn) \varepsilon(mn + mh) e^{-2\pi i \alpha m(n+h)} \right| \ll X^{1-\varkappa}.$$

Choose a positive integer k from the inequalities

$$2^{k-1} < MX^{2\rho} \leq 2^k.$$

Introduce the symbol $\varepsilon_k(n)$:

$$\varepsilon_k(n) = \begin{cases} 1, & \text{if the sum of the first } k \text{ binary digits of } n \text{ is even;} \\ -1, & \text{otherwise.} \end{cases}$$

Prove that

$$|W_4(M, N)| \ll \sum_{N/2 < n \leq N} \left| \sum_{M/2 < m \leq M_1} \varepsilon_k(mn) \varepsilon_k(mn + mh) e^{-2\pi i \alpha m(n+h)} \right| + X^{1-\rho+\varepsilon}.$$

Divide mn by 2^k with remainder: $mn = 2^k q + r$, $0 \leq r < 2^k$. If $r < 2^k - 2MH$, then $mn + mh = 2^k q + r + mh$, $0 < r + mh < 2^k$.

For such mn , we have

$$\varepsilon(mn) \varepsilon(mn + mh) = \varepsilon(r) \varepsilon(r + mh) = \varepsilon_k(mn) \varepsilon_k(mn + mh).$$

The number of pairs (m, n) such that the remainder r lies between $2^k - 2MH$ and $2^k - 1$ is $O(X2^{-k}MHX^\varepsilon) = O(X^{1-\rho+\varepsilon})$.

Now we estimate

$$|W_5(M, N)| = \sum_{N/2 < n \leq N} \left| \sum_{M/2 < m \leq M_1} \varepsilon_k(mn) \varepsilon_k(mn + mh) e^{-2\pi i \alpha hm} \right|.$$

Introduce the discrete Fourier transform for the character $\varepsilon_k(r)$:

$$\hat{\varepsilon}_k(r) = 2^{-k} \sum_{l=0}^{2^k-1} \varepsilon_k(l) e^{-2\pi i \frac{rl}{2^k}}.$$

From this definition it follows that

$$\varepsilon_k(mn) = \sum_{r=0}^{2^k-1} \hat{\varepsilon}_k(r) \exp\left\{\frac{2\pi i r mn}{2^k}\right\}, \quad \varepsilon_k(mn + mh) = \sum_{s=0}^{2^k-1} \hat{\varepsilon}_k(s) \exp\left\{\frac{2\pi i s(mn + mh)}{2^k}\right\}.$$

Summing linear sums over m , we obtain:

$$|W_5(M, N)| \leq \sum_{r=0}^{2^k-1} \sum_{s=0}^{2^k-1} |\hat{\varepsilon}_k(r)| |\hat{\varepsilon}_k(s)| \sum_{N/2 < n \leq N} \min\left(M, \left\| \frac{r+s}{2^k} n + \frac{hs}{2^k} - h\alpha \right\|^{-1}\right).$$

From now on we will assume that $\rho = \frac{c}{200}$, where $c = \frac{1-\theta_0}{2}$, θ_0 is a constant from lemma 4. Let t – non-negative integer such that $2^t \|(r+s)$.

Suppose first $0 \leq t \leq k - \frac{2\rho}{c} \log_2 X$. Note that from inequality $2^k > M \geq X^{1/10}$, it follows that $k > \frac{1}{10} \log_2 X$; this and $\frac{2\rho}{c} = \frac{1}{100}$ implies the inequality $k - \frac{2\rho}{c} \log_2 X \geq \frac{4}{5}k$.

We apply Lemma 1 with $q = 2^{k-t}$, $\alpha = \frac{r+s}{2^k}$, $\beta = \frac{sh}{2^k} - \alpha h$ to the sum

$$\sum_{N/2 < n \leq N} \min \left(M, \left\| \frac{r+s}{2^k} n + \frac{sh}{2^k} - \alpha h \right\|^{-1} \right) : \\ \sum_{N/2 < n \leq N} \min \left(M, \left\| \frac{r+s}{2^k} n + \frac{sh}{2^k} - \alpha h \right\|^{-1} \right) \ll \left(\frac{N}{2^{k-t}} + 1 \right) (M + k2^{k-t}).$$

Simplify the right-hand side of this inequality.

We have:

$$\frac{N}{2^{k-t}} + 1 < \frac{NX^{2\rho}}{2^{k-t}} + 1 \leq 2 \frac{NX^{2\rho}}{2^{k-t}}; \\ M + k2^{k-t} < MX^{2\rho} + k2^{k-t} < 2MX^{3\rho};$$

and so,

$$\left(\frac{N}{2^{k-t}} + 1 \right) (M + k2^{k-t}) < 4MNX^{5\rho} 2^{-(k-t)} \leq 5X^{1+5\rho} 2^{-(k-t)}, \\ \sum_{N/2 < n \leq N} \min \left(M, \left\| \frac{r+s}{2^k} n + \frac{sh}{2^k} - \alpha h \right\|^{-1} \right) \ll X^{1+5\rho} 2^{-(k-t)}.$$

Now estimate the sum

$$\sum_{\substack{r=0 \\ 2^t \parallel (r+s)}}^{2^k-1} \sum_{s=0}^{2^k-1} |\hat{\varepsilon}_k(r)| |\hat{\varepsilon}_k(s)| \leq \sum_{\substack{a=0 \\ r \equiv a}}^{2^t-1} \sum_{\substack{r=0 \\ (\bmod 2)^k}}^{2^k-1} |\hat{\varepsilon}_k(r)| \sum_{\substack{s=0 \\ s \equiv -a \\ (\bmod 2)^k}}^{2^k-1} |\hat{\varepsilon}_k(s)|.$$

Use the corollary 4:

$$\sum_{\substack{r=0 \\ 2^t \parallel (r+s)}}^{2^k-1} \sum_{s=0}^{2^k-1} |\hat{\varepsilon}_k(r)| |\hat{\varepsilon}_k(s)| \ll 2^{(1-2c)(k-t)} \log^2 X \sum_{a=0}^{2^k-1} |\hat{\varepsilon}_k(a)|^2 = 2^{(1-2c)(k-t)} \ln^2 X.$$

In this case, the estimate is achieved

$$|W_5(M, N)| \ll X^{1+6\rho} 2^{-2c(k-t)} \ln^2 X.$$

Recall that

$$k - t \geq \frac{4}{5}k, \quad 2^k > M \geq X^{1/10},$$

it follows that

$$|W_5(M, N)| \ll X^{1+6\rho - \frac{8}{50}c};$$

finally, from the inequality $\rho \leq \frac{c}{100}$ we have $6\rho - \frac{8}{50}c < -\rho$,

$$|W_5(M, N)| \ll X^{1-\rho}.$$

It remains to consider the case

$$k - \frac{2\rho}{c} \log_2 X < t \leq k.$$

We have:

$$\begin{aligned}
|W_5(M, N)| &\leq \sum_{\substack{r=0 \\ 2^t \parallel (r+s)}}^{2^k-1} \sum_{s=0}^{2^k-1} |\hat{\varepsilon}_k(r)| |\hat{\varepsilon}_k(s)| \min \left(M, \left\| \frac{r+s}{2^k} n + \frac{sh}{2^k} - \alpha h \right\|^{-1} \right) = \\
&= \sum_{N/2 < n \leq N} \sum_{s_2=0}^{2^{k-t}-1} \sum_{r_2=0}^{2^{k-t}-1} \sum_{\substack{s_1=0 \\ r_1+s_1 \equiv 0 \pmod{2^t}}}^{2^t-1} \sum_{r_1=0}^{2^k-1} |\hat{\varepsilon}_k(r_1 + 2^t r_2)| |\hat{\varepsilon}_k(s_1 + 2^t s_2)| \times \\
&\quad \times \min \left(M, \left\| \frac{r_1 + s_1 + 2^t(r_2 + s_2)}{2^k} n + \frac{(s_1 + 2^t s_2)h}{2^k} - \alpha h \right\|^{-1} \right).
\end{aligned}$$

It follows from the inequalities $0 \leq s_1; r_1 < 2^t$ and congruence $r_1 + s_1 \equiv 0 \pmod{2^t}$ that either $r_1 = s_1 = 0$ or $r_1 + s_1 = 2^t$.

From this and Lemma 2 we get

$$\begin{aligned}
|W_5(M, N)| &\ll 2^{-2k(1-\lambda)} \sum_{N/2 < n \leq N} \sum_{s_2=0}^{2^{k-t}-1} \sum_{r_2=0}^{2^{k-t}-1} \sum_{s_1=0}^{2^k-1} \min \left(M, \left\| \frac{hs_1}{2^k} + \beta \right\|^{-1} \right) + \\
&\quad + X 2^{-2k(1-\lambda)} 2^{2(k-t)},
\end{aligned} \tag{1}$$

where $\beta = \frac{1+r_2+s_2}{2^{k-t}} n + \frac{hs_2}{2^{k-t}} - \alpha h$.

Let $\frac{hs_1}{2^k} = \frac{h_1 s_1}{2^{k_1}}$, where $(h_1 s_1, 2) = 1$. From Lemma 1 and inequalities

$$2^{-k_1} \leq 2^{-k} X^\rho, \quad M + 2^{k_1} k_1 \ll M X^{3\rho}$$

it follows that

$$\sum_{s_1=0}^{2^k-1} \min \left(M, \left\| \frac{hs_1}{2^k} + \beta \right\|^{-1} \right) \ll M X^{5\rho}.$$

Substituting this inequality in (1):

$$|W_5(M, N)| \ll X^{1+5\rho} 2^{k-t} 2^{-2k(1-\lambda)}.$$

Now use the fact that

$$2^{k-t} \leq X^{4\rho/c}, \quad \frac{\rho}{c} \leq \frac{1}{200}, \quad c < 0.06, \quad 2^k > M \geq X^{0.1}.$$

We got:

$$|W_5(M, N)| \ll X^{1-0.01}.$$

Theorem 1 is proved.

5. Proof of Theorem 2

Define sums $S(\alpha)$ and $S_0(\alpha)$:

$$S(\alpha) = \sum_{p \leq N} e^{2\pi i \alpha p}, \quad S_0(\alpha) = \sum_{p \leq N} \varepsilon(p) e^{2\pi i \alpha p}.$$

Then

$$J_0(N) = \frac{1}{8} \int_0^1 (S(\alpha) + S_0(\alpha))^3 e^{-2\pi i \alpha N} d\alpha.$$

Expanding the brackets and using Theorem 1 and Cauchy's inequality, we obtain

$$J_0(N) = \frac{1}{8} \int_0^1 S^3(\alpha) e^{-2\pi i \alpha N} d\alpha + O(\pi(N) N^{1-\varepsilon}).$$

Since

$$J(N) = \int_0^1 S^3(\alpha) e^{-2\pi i N \alpha} d\alpha, \quad J(N) \gg N^2 (\ln N)^{-3}$$

(for sufficiently large odd N), theorem 2 is proved.

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